

2014年 東大数学 理系第3問 ②

$$(3) \int_{-1-\sqrt{3}}^{-1+\sqrt{3}} (u^2+u+1) \sqrt{-u^2-2u+2} du$$

を求める。

$$= \int_{-1-\sqrt{3}}^{-1+\sqrt{3}} (u^2+u+1) \sqrt{-(u+1)^2+3} du$$

$\sqrt{r^2 - x^2} dx$  は  
 $x = r \sin \theta$  と置換。

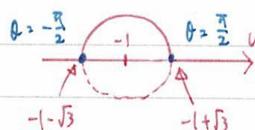
$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (3\sin^2 \theta + \sqrt{3}\sin \theta + 1) \sqrt{-3\sin^2 \theta + 3} \times \sqrt{3}\cos \theta d\theta$$

$$\begin{aligned} u &= -1-\sqrt{3} \rightarrow -1+\sqrt{3} \\ \theta &= -\frac{\pi}{2} \rightarrow \frac{\pi}{2} \end{aligned}$$

$$du = 3\cos \theta d\theta$$

$$v = \sqrt{-(u+1)^2 + 3}$$

$$(u+1)^2 + v^2 = 3$$



$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (3\sin^2 \theta + \sqrt{3}\sin \theta + 1) \sqrt{3\cos^2 \theta} \cdot \sqrt{3}\cos \theta d\theta$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (3\sin^2 \theta + \sqrt{3}\sin \theta + 1) \cdot \sqrt{3}\cos \theta \cdot \sqrt{3}\cos \theta \cdot d\theta$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (3\sin^2 \theta + \sqrt{3}\sin \theta + 1) \cdot 3\cos^2 \theta \cdot d\theta \cdot \int_{-a}^a (\text{偶関数}) dx$$

$$= 2 \int_0^a (\text{偶関数}) dx$$

$$\cdot \int_{-a}^a (\text{奇関数}) dx$$

= 0 を利用。

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (9\sin^2 \theta \cos^2 \theta + 3\sqrt{3}\sin \theta \cos^2 \theta + 3\cos^2 \theta) d\theta$$

$$= 2 \int_0^{\frac{\pi}{2}} (9\sin^2 \theta \cos^2 \theta + 3\cos^2 \theta) d\theta$$

$$= 18 \int_0^{\frac{\pi}{2}} \sin^2 \theta \cos^2 \theta d\theta + 6 \int_0^{\frac{\pi}{2}} \cos^2 \theta d\theta$$

次数下げ？

$$\int_0^{\frac{\pi}{2}} \sin^2 \theta \cos^2 \theta d\theta = \int_0^{\frac{\pi}{2}} \left(\frac{1}{2}\sin 2\theta\right)^2 d\theta \quad \sin 2\theta = 2\sin \theta \cos \theta \Rightarrow \sin \theta \cos \theta = \frac{1}{2}\sin 2\theta$$

$$= \frac{1}{4} \int_0^{\frac{\pi}{2}} \sin^2 2\theta d\theta = \frac{1}{4} \int_0^{\frac{\pi}{2}} \frac{1 - \cos 4\theta}{2} d\theta \quad \sin^2 \theta = \frac{1 - \cos 2\theta}{2}$$

$$= \frac{1}{4} \left[ \frac{1}{2}\theta - \frac{1}{8}\sin 4\theta \right]_0^{\frac{\pi}{2}} = \frac{1}{4} \left[ \left( \frac{1}{2} \times \frac{\pi}{2} - \frac{1}{8}\sin 2\pi \right) - (0 - 0) \right] = \frac{\pi}{16}$$

$$\int_0^{\frac{\pi}{2}} \cos^2 \theta d\theta = \int_0^{\frac{\pi}{2}} \frac{1 + \cos 2\theta}{2} d\theta \quad \cos^2 \theta = \frac{1 + \cos 2\theta}{2}$$

$$= \left[ \frac{1}{2}\theta + \frac{1}{4}\sin 2\theta \right]_0^{\frac{\pi}{2}} = \frac{\pi}{4}$$

以上。)

$$18 \times \frac{\pi}{16} + 6 \times \frac{\pi}{4}$$

$$= \frac{9}{8}\pi + \frac{12}{8}\pi$$

$$= \frac{21}{8}\pi$$